Closed Form Solution of the Exterior-Point Eshelby Tensor for an Elliptic Cylindrical Inclusion

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With the help of the I-integrals expressed by Mura (1987, Micromechanics of Defects in Solids, 2nd ed., Martinus Nijhoff, Dordrecht) and the outward unit normal vector introduced by Ju and Sun (1999, "A Novel Formulation for the Exterior-Point Eshelby's Tensor of an Ellipsoidal Inclusion," ASME Trans. J. Appl. Mech., ***66****, pp. 570–574), the closed form solution of the exterior-point Eshelby tensor for an elliptic cylindrical inclusion is derived in this work. The proposed closed form of the Eshelby tensor for an elliptic cylindrical inclusion is more explicit than that given by Mura, which is rough and unfinished. The Eshelby tensor for an elliptic cylindrical inclusion can be reduced to the Eshelby tensor for a circular cylindrical inclusion by letting the aspect ratio of the inclusion $\alpha=1$. The closed form Eshelby tensor presented in this study can contribute to micromechanics-based analysis of composites with elliptic cylindrical inclusions.*
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1 Introduction

A micromechanical approach to the analysis of composites has the advantage of establishing a suitable micromechanical model that enables the effective properties, including damage evolution, to be predicted through an understanding of the fundamental concepts of composites [1,2]. In micromechanics, the microstructures of composites are simplified in characterization models on the basis of homogenization theories, and those theories are based on the type of modeling that transforms the body of a heterogeneous material into a constitutive equivalent body of a homogeneous continuum [3]. Among the solutions of the inclusion and inhomogeneity problems of composites, Eshelby's equivalent inclusion method [4,5] is a convenient solution for inhomogeneities and an indispensable part of the theoretical foundation of modern composite mechanics [6–9].

The Eshelby tensor, which relates the strain field in the inclusion to the prescribed uniform eigenstrain, is required in Eshelby's equivalent inclusion method. In Eshelby's inclusion problem [4,5], one considers an infinite homogeneous medium D consisting of a matrix $D-\Omega$ and an inclusion Ω embedded in a matrix that has the same constitutive material of the matrix. Let us suppose that the uniform eigenstrain tensor $\boldsymbol{\epsilon}^*$, which is also called a stress-free transformation strain, is prescribed and the homogeneous body D is subjected to a far-field applied loading. To describe the strain and stress field in that inclusion problem, Eshelby [4,5] proposed a fourth-rank tensor, which is referred to as an Eshelby tensor. The Eshelby tensor is divided into two different expressions. The expressions depend on the location of the material point $\mathbf{x}$, which refers to the desired point of the strain and stress field. When $\mathbf{x}$ is inside the inclusion, the tensor is called an interior-point Eshelby tensor $\mathbf{S}$; and when $\mathbf{x}$ is outside the inclusion, the tensor is called an exterior-point Eshelby tensor $\hat{\mathbf{G}}(\mathbf{x})$ [10].

Although many researchers have used the Eshelby tensor for inclusions of various shapes, the task of applying it to composites with unidirectional inclusion is difficult. Mura [11] presented a detailed analytical expression of the Eshelby tensor; however, the analytical solution for the interior-point Eshelby tensor is focused while that for the exterior-point Eshelby tensor is rough and unfinished. Note that no studies of the closed form of the exterior-point Eshelby tensor for an elliptic cylindrical inclusion have been reported in literature. Hence, the present work attempts to derive the closed form solution of the exterior-point Eshelby tensor for an elliptic cylindrical inclusion. The *I*-integrals expressed by Mura [11] and the outward unit normal vector $\hat{\mathbf{n}}$ introduced by Ju and Sun [10] are used in the derivation. As a special case, the Eshelby tensor for an elliptic cylindrical inclusion can be reduced to the Eshelby tensor for a circular cylindrical inclusion.

2 The Eshelby Tensor for an Elliptic Cylindrical Inclusion

2.1 Recapitulation of the Analytical Expression of the Eshelby Tensor for an Elliptic Cylindrical Inclusion. Let us consider an elliptic cylindrical inclusion embedded in an isotropic infinite body, as shown in Fig. 1. According to the Cartesian coordinate system, the domain of an elliptic cylindrical inclusion can be expressed as (cf. Ref. [11])

$$\frac{x_i x_i}{a_I^2} \le 1 \quad (i=1,2) \quad \text{and} \quad a_3 \to \infty \tag{1}$$

where a_I $(I=1,2)$ is the radius of the cross section of the elliptic cylinder and the subscripts i and I follow the summation convention mentioned in Mura [11]: That is, the repeated lowercase indices are summed from 1 to 2 and the uppercase indices take on the same numbers as the corresponding lowercase indices but are not summed.

As in the studies of Mura [11] and Ju and Sun [10], if point $\mathbf{x}$ is located outside the inclusion, an imaginary ellipse is created. The ellipse, which is shown in Fig. 1, can be expressed as

$$\frac{x_i x_i}{a_I^2+\lambda} = 1 \quad (i=1,2) \tag{2}$$

where λ is the positive root of Eq. (2). Ju and Sun [10] also introduced the following expression for the outward unit normal vector $\hat{\mathbf{n}}$ at any matrix point $\mathbf{x}$ on the imaginary ellipse:

$$\hat{n}_i = \frac{\frac{x_i}{a_I^2+\lambda}}{\sqrt{\frac{x_j x_j}{(a_J^2+\lambda)^2}}} \tag{3}$$

As in Mura [11], the strain field in the inclusion problem can be written in the form

$$\epsilon_{ij}(\mathbf{x}) = \hat{G}_{ijkl}(\mathbf{x})\epsilon_{kl}^* \tag{4}$$

where ϵ_{kl}^* is the eigenstrain tensor and $\hat{G}_{ijkl}$ is the exterior-point Eshelby tensor and can be expressed as [11]

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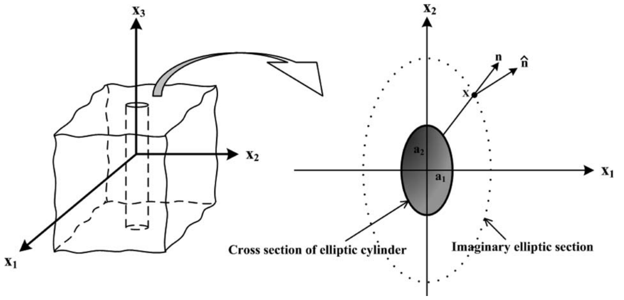

Fig. 1 A schematic description of an elliptic cylindrical inclusion embedded in an isotropic infinite body (cf. Ref. [2])

$$\hat{G}_{ijkl}(\mathbf{x}) = \frac{1}{8\pi(1-\nu)}[\psi_{,klij} - 2\nu\delta_{kl}\phi_{,ij} - (1-\nu)\{\phi_{,kj}\delta_{il} + \phi_{,ki}\delta_{jl} + \phi_{,lj}\delta_{ik} + \phi_{,li}\delta_{jk}\}] \tag{5}$$

with

$$\psi(\mathbf{x}) = \int_{\Omega} |\mathbf{x}-\mathbf{x}'| d\mathbf{x}', \quad \phi(\mathbf{x}) = \int_{\Omega} 1/|\mathbf{x}-\mathbf{x}'| d\mathbf{x}' \tag{6}$$

in which $\mathbf{x}'$ is the local point inside the inclusion domain and δ_{ij} signifies the Kronecker delta. In addition, $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ can also be written in the form of I-integrals as [11]

$$\phi(\mathbf{x}) = \tfrac{1}{2}[I(\lambda) - x_n x_n I_N(\lambda)] \tag{7}$$

$$\psi_{,i}(\mathbf{x}) = \frac{x_i}{2}\{I(\lambda) - x_n x_n I_N(\lambda) - a_I^2[I(\lambda) - x_n x_n I_{IN}(\lambda)]\} \tag{8}$$

Using Eqs. (7) and (8), the exterior-point Eshelby tensor $\hat{G}_{ijkl}$ in Eq. (5) can be recast as [11]

$$\begin{aligned}\hat{G}_{ijkl}(\mathbf{x}) = {}& \frac{1}{8\pi(1-\nu)}[8\pi(1-\nu)S_{ijkl}(\lambda) + 2\nu\delta_{kl}x_i I_{I,j}(\lambda) + (1-\nu) \\ & \times\{\delta_{il}x_k I_{K,j}(\lambda) + \delta_{jl}x_k I_{K,i}(\lambda) + \delta_{ik}x_l I_{L,j}(\lambda) + \delta_{jk}x_l I_{L,i}(\lambda)\} \\ & - \delta_{ij}x_k[I_K(\lambda) - a_I^2 I_{KI}(\lambda)]_{,l} - (\delta_{ik}x_j + \delta_{jk}x_i)[I_J(\lambda) \\ & - a_I^2 I_{IJ}(\lambda)]_{,l} - (\delta_{il}x_j + \delta_{jl}x_i)[I_J(\lambda) - a_I^2 I_{IJ}(\lambda)]_{,k} - x_i x_j[I_J(\lambda) \\ & - a_I^2 I_{IJ}(\lambda)]_{,lk}]\end{aligned} \tag{9}$$

with

$$\begin{aligned}S_{ijkl}(\lambda) = {}& \frac{1}{8\pi(1-\nu)}[\delta_{ij}\delta_{kl}\{2\nu I_I(\lambda) - I_K(\lambda) + a_I^2 I_{KI}(\lambda)\} + (\delta_{ik}\delta_{jl} \\ & + \delta_{il}\delta_{jk})\{a_I^2 I_{IJ}(\lambda) - I_J(\lambda) + (1-\nu)[I_K(\lambda) + I_L(\lambda)]\}]\end{aligned} \tag{10}$$

where ν denotes Poisson's ratio of the matrix, and details of the I-integrals in Eqs. (7) and (8) can be seen in Mura [11].

2.2 Closed Form Solution of the Eshelby Tensor for an Elliptic Cylindrical Inclusion. Section 2.1 introduced the rough form of the exterior-point Eshelby tensor for an elliptic cylindrical inclusion mentioned in Mura [11]. However, there is a need for a compact expansion of the Eshelby tensor for an elliptic cylindrical inclusion because the expression of Mura [11] for an elliptic cylindrical inclusion is unfinished. For the closed form of the Eshelby tensor, the I-integrals and their derivatives in Eq. (9) are newly derived in terms of an outward unit normal vector $\hat{\mathbf{n}}$ in this study.

The derivative of λ is given by [11]

$$\lambda_{,i} = \frac{\frac{2x_i}{a_I^2+\lambda}}{\frac{x_k x_k}{(a_k^2+\lambda)^2}} \tag{11}$$

Alternatively, from Eqs. (3) and (11), the first derivative of λ can be written in terms of the outward normal vector $\hat{\mathbf{n}}$ as

$$\lambda_{,i} = \frac{2}{\sqrt{\frac{x_k x_k}{(a_k^2+\lambda)^2}}}\hat{n}_i \tag{12}$$

Furthermore, the second derivative of λ can also be derived from the first derivative of λ as

$$\begin{aligned}\lambda_{,ij} = {}& \frac{2(a_K^2+\lambda)^2}{(a_I^2+\lambda)}\left[\frac{\delta_{ji}}{x_k x_k} - \frac{2x_i\delta_{jk}}{x_k{}^3}\right] + \frac{8x_i(a_K^2+\lambda)^2}{(x_k x_k)^{3/2}(a_I^2+\lambda)}\left[1 \right. \\ & \left. - \frac{a_K^2+\lambda}{2(a_I^2+\lambda)}\right]\hat{n}_j\end{aligned} \tag{13}$$

The I-integrals in Eqs. (9) and (10) for an elliptic cylinder ($a_3 \to \infty$) are given by [11]

$$I_1(\lambda) = \frac{4\pi a_1 a_2}{a_2^2 - a_1^2}\left\{\frac{\sqrt{a_2^2+\lambda}}{\sqrt{a_1^2+\lambda}} - 1\right\},$$

$$I_2(\lambda) = \frac{4\pi a_1 a_2}{a_1^2 - a_2^2}\left\{\frac{\sqrt{a_1^2+\lambda}}{\sqrt{a_2^2+\lambda}} - 1\right\}, \quad I_3(\lambda) = 0 \tag{14}$$

In addition, the integral I_{IJ} is given by [11]

$$I_{11}(\lambda) = \frac{1}{3}\left[\frac{4\pi a_1 a_2 a_3}{(a_1^2+\lambda)\Delta(\lambda)} - I_{12}(\lambda) - I_{13}(\lambda)\right] \tag{15}$$

$$I_{12}(\lambda) = \frac{I_2(\lambda) - I_3(\lambda)}{a_1^2 - a_2^2} \tag{16}$$

with

$$\Delta(\lambda) = \sqrt{(a_1^2+\lambda)(a_2^2+\lambda)(a_3^2+\lambda)} \tag{17}$$

where I_{IJ} ($I, J = 1, 2, 3$) follows the cyclic permutation with respect to 1, 2, and 3 as described in Mura [11].

With the help of the definitions in Eqs. (12)–(17), the nonzero components of the exterior-point Eshelby tensor $\hat{\mathbf{G}}(\mathbf{x})$ in Eq. (9) can be *explicitly* derived as follows.

(i) In the longitudinal direction (3-axis),

$$\hat{G}_{1133} = \frac{\alpha\omega\rho^2\nu}{1-\nu}\left[\frac{1}{1+\omega} - \hat{n}_1\hat{n}_1\right] \tag{18}$$

$$\hat{G}_{2233} = \frac{\alpha\omega\rho^2\nu}{1-\nu}\left[\frac{\omega}{1+\omega} - \hat{n}_2\hat{n}_2\right] \tag{19}$$

$$\hat{G}_{1313} = \frac{\alpha\omega\rho^2}{1-\nu}\left[\frac{1}{1+\omega} - \hat{n}_1\hat{n}_1\right] \tag{20}$$

$$\hat{G}_{2323} = \frac{\alpha\omega\rho^2}{1-\nu}\left[\frac{\omega}{1+\omega} - \hat{n}_2\hat{n}_2\right] \tag{21}$$

$$\hat{G}_{i3j3} = \hat{G}_{i33j} = \hat{G}_{3ij3} = \hat{G}_{3i3j} \tag{22}$$

with

$$\omega = \frac{1}{\sqrt{(\alpha^2-1)\rho^2+1}}, \quad \alpha = \frac{a_2}{a_1}, \quad \rho = \frac{a_1}{\sqrt{a_1+\lambda}} \tag{23}$$

(ii) In the transverse direction (1-, 2-axis),

(a) for components with no $\hat{n}_i$,

$$\hat{G}_{1111}=\frac{\alpha\rho^2}{2(1-\nu)}[\xi(1-2\nu)+\omega\rho^2(1-\xi^2)] \tag{24}$$

$$\hat{G}_{2222}=\frac{\alpha\rho^2}{2(1-\nu)}[\omega\xi(1-2\nu)+\alpha^2\omega\rho^2(\omega^2-\xi^2)] \tag{25}$$

$$\hat{G}_{1122}=\frac{\alpha\rho^2}{2(1-\nu)}[\xi(2\nu-\omega)+\omega\rho^2\xi^2] \tag{26}$$

$$\hat{G}_{2211}=\frac{\alpha\rho^2}{2(1-\nu)}[\xi(\omega\cdot 2\nu-1)+\alpha^2\omega\rho^2\xi^2] \tag{27}$$

$$\hat{G}_{1212}=\hat{G}_{1221}=\frac{\alpha\rho^2}{2(1-\nu)}[\xi\cdot\{(1-\nu)-\omega\nu\}+\omega\rho^2\xi^2] \tag{28}$$

$$\hat{G}_{2112}=\hat{G}_{2121}=\frac{\alpha\rho^2}{2(1-\nu)}[\xi\cdot\{(1-\nu)\omega-\nu\}+\alpha^2\omega\rho^2\xi^2] \tag{29}$$

with

$$\xi=\frac{\omega}{1+\omega} \tag{30}$$

(b) for components with $\hat{n}_i\hat{n}_j$,

$$\hat{G}_{1111}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[2\nu-\frac{4}{3}\{(3+\omega^2)\rho^2+\Psi_1\}\right]\hat{n}_1\hat{n}_1 \tag{31}$$

$$\hat{G}_{2222}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[2\nu-\frac{4}{3}\left\{(1+3\omega^2)\alpha^2\rho^2+\frac{\alpha^2}{\omega^2}\Psi_1\right\}\right]\hat{n}_2\hat{n}_2 \tag{32}$$

$$\hat{G}_{1112}=\frac{\alpha\rho^2\omega}{2(1-\nu)}[1+2\nu-\{(3+\omega^2)\rho^2+\Psi_1\}]\hat{n}_1\hat{n}_2 \tag{33}$$

$$\hat{G}_{2221}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[1+2\nu-\left\{(1+3\omega^2)\alpha^2\rho^2+\frac{\alpha^2}{\omega^2}\Psi_1\right\}\right]\hat{n}_2\hat{n}_1 \tag{34}$$

$$\hat{G}_{1121}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[1+2\nu+\frac{1}{\omega^2}\Psi_1-\frac{2}{3}\{(3+\omega^2)\rho^2+\Psi_1\}\right]\hat{n}_2\hat{n}_1 \tag{35}$$

$$\hat{G}_{2212}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[1+2\nu+\alpha^2\Psi_1-\frac{2}{3}\left\{(1+3\omega^2)\alpha^2\rho^2+\frac{\alpha^2}{\omega^2}\Psi_1\right\}\right]\hat{n}_1\hat{n}_2 \tag{36}$$

$$\hat{G}_{1211}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-1+\frac{1}{\omega^2}\Psi_1\right]\hat{n}_1\hat{n}_2 \tag{37}$$

$$\hat{G}_{2122}=\frac{\alpha\rho^2\omega}{2(1-\nu)}[-1+\alpha^2\Psi_1]\hat{n}_2\hat{n}_1 \tag{38}$$

$$\hat{G}_{1222}=\frac{\alpha\rho^2\omega}{2(1-\nu)}[-2+\omega^2+\Psi_1]\hat{n}_1\hat{n}_2 \tag{39}$$

$$\hat{G}_{2111}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-2+\frac{1}{\omega^2}+\frac{\alpha^2}{\omega^2}\Psi_1\right]\hat{n}_2\hat{n}_1 \tag{40}$$

$$\hat{G}_{1122}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[\left\{-2\nu-\omega^2+\frac{1}{3}\omega^2[(3+\omega^2)\rho^2+\Psi_1]\right\}\hat{n}_1\hat{n}_1+\left\{1+\frac{1}{\omega^2}\Psi_1\right\}\hat{n}_2\hat{n}_2\right] \tag{41}$$

$$\hat{G}_{2211}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[\left\{-2\nu-\omega^2+\frac{1}{3}\cdot\frac{\alpha^2}{\omega^2}\left[(1+3\omega^2)\rho^2+\frac{1}{\omega^2}\Psi_1\right]\right\}\hat{n}_2\hat{n}_2+\{1+\alpha^2\Psi_1\}\hat{n}_1\hat{n}_1\right] \tag{42}$$

$$\hat{G}_{1212}=\hat{G}_{1221}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[\{-1+\nu+\omega^2+\Psi_1\}\hat{n}_1\hat{n}_1+\left\{\nu+\frac{1}{\omega^2}\Psi_1\right\}\hat{n}_2\hat{n}_2\right] \tag{43}$$

$$\hat{G}_{2112}=\hat{G}_{2121}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[\left\{-1+\nu+\frac{1}{\omega^2}+\frac{\alpha^2}{\omega^2}\Psi_1\right\}\hat{n}_2\hat{n}_2+\{\nu+\alpha^2\Psi_1\}\hat{n}_1\hat{n}_1\right] \tag{44}$$

with

$$\Psi_1=\frac{1}{\rho^2(1-\alpha^2)^2}\left(2-\omega^2-\frac{1}{\omega^2}\right) \tag{45}$$

(c) for components with $\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_l$,

$$\hat{G}_{1111}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-(5+\omega^2)+\frac{1}{3}(21+8\omega^2+3\omega^4)\rho^2+\frac{2}{3}\Psi_1+\Psi_2\right]\hat{n}_1\hat{n}_1\hat{n}_1\hat{n}_1 \tag{46}$$

$$\hat{G}_{2222}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-\left(5+\frac{1}{\omega^2}\right)+\frac{\alpha^2\omega^2}{3}\left(21+\frac{8}{\omega^2}+\frac{3}{\omega^4}\right)\rho^2+\frac{\alpha^2}{\omega^4}\left(\frac{2}{3}\omega^2\Psi_1+\Psi_2\right)\right]\hat{n}_2\hat{n}_2\hat{n}_2\hat{n}_2 \tag{47}$$

$$\hat{G}_{1112}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3(1+\omega^2)+\frac{1}{3}(15+12\omega^2+5\omega^4)\rho^2+\frac{2}{3}\omega^2\Psi_1+\Psi_2\right]\hat{n}_1\hat{n}_1\hat{n}_1\hat{n}_2 \tag{48}$$

$$\hat{G}_{2221}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3\left(1+\frac{1}{\omega^2}\right)+\frac{\alpha^2\omega^2}{3}\left(15+\frac{12}{\omega^2}+\frac{5}{\omega^4}\right)\rho^2+\frac{\alpha^2}{\omega^4}\left(\frac{2}{3}\Psi_1+\Psi_2\right)\right]\hat{n}_2\hat{n}_2\hat{n}_2\hat{n}_1 \tag{49}$$

$$\hat{G}_{1121}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-(5+\omega^2)+\frac{1}{3}(15+12\omega^2+5\omega^4)\rho^2+\frac{2}{3}\omega^2\Psi_1+\Psi_2\right]\hat{n}_1\hat{n}_1\hat{n}_2\hat{n}_1 \tag{50}$$

$$\hat{G}_{2212}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-\left(5+\frac{1}{\omega^2}\right)+\frac{\alpha^2\omega^2}{3}\left(15+\frac{12}{\omega^2}+\frac{5}{\omega^4}\right)\rho^2+\frac{\alpha^2}{\omega^4}\left(\frac{2}{3}\Psi_1+\Psi_2\right)\right]\hat{n}_2\hat{n}_2\hat{n}_1\hat{n}_2 \quad (51)$$

$$\hat{G}_{1122}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3(1+\omega^2)+\frac{1}{3}(15+12\omega^2+5\omega^4)\rho^2+\frac{2}{3}\omega^2\Psi_1+\Psi_2\right]\hat{n}_1\hat{n}_1\hat{n}_2\hat{n}_2 \quad (52)$$

$$\hat{G}_{2211}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3\left(1+\frac{1}{\omega^2}\right)+\frac{\alpha^2\omega^2}{3}\left(15+\frac{12}{\omega^2}+\frac{5}{\omega^4}\right)\rho^2+\frac{\alpha^2}{\omega^4}\left(\frac{2}{3}\Psi_1+\Psi_2\right)\right]\hat{n}_2\hat{n}_2\hat{n}_1\hat{n}_1 \quad (53)$$

$$\hat{G}_{1211}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3(1+\omega^2)-\left(\frac{2}{\omega^2}\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_1\hat{n}_2\hat{n}_1\hat{n}_1 \quad (54)$$

$$\hat{G}_{2122}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3\left(1+\frac{1}{\omega^2}\right)-\alpha^2\left(2\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_2\hat{n}_1\hat{n}_2\hat{n}_2 \quad (55)$$

$$\hat{G}_{1212}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-(1+5\omega^2)-\left(2\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_1\hat{n}_2\hat{n}_1\hat{n}_2 \quad (56)$$

$$\hat{G}_{2121}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-\left(1+\frac{5}{\omega^2}\right)-\alpha^2\left(\frac{2}{\omega^2}\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_2\hat{n}_1\hat{n}_2\hat{n}_1 \quad (57)$$

$$\hat{G}_{1221}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3(1+\omega^2)-\left(\frac{2}{\omega^2}\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_1\hat{n}_2\hat{n}_2\hat{n}_1 \quad (58)$$

$$\hat{G}_{2112}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-3\left(1+\frac{1}{\omega^2}\right)-\alpha^2\left(2\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_2\hat{n}_1\hat{n}_1\hat{n}_2 \quad (59)$$

$$\hat{G}_{1222}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-(1+5\omega^2)-\left(2\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_1\hat{n}_2\hat{n}_2\hat{n}_2 \quad (60)$$

$$\hat{G}_{2111}=\frac{\alpha\rho^2\omega}{2(1-\nu)}\left[-\left(1+\frac{5}{\omega^2}\right)-\alpha^2\left(\frac{2}{\omega^2}\Psi_1+\frac{3}{\omega^2}\Psi_2\right)\right]\hat{n}_2\hat{n}_1\hat{n}_1\hat{n}_1 \quad (61)$$

with

$$\Psi_2=\frac{1}{\rho^2(1-\alpha^2)^2}\left(1+\omega^2-\omega^4-\frac{1}{\omega^2}\right) \quad (62)$$

The components of the interior-point Eshelby tensor $\mathbf{S}$ can be derived by letting $\hat{n}_1=\hat{n}_2=0$ and $\rho=1$ in the components of the exterior-point Eshelby tensor $\hat{\mathbf{G}}(\mathbf{x})$ and is expressed as

$$S_{1111}=\frac{1}{2(1-\nu)}\cdot\frac{\alpha}{1+\alpha}\left[1-2\nu+\frac{2+\alpha}{1+\alpha}\right] \quad (63)$$

$$S_{2222}=\frac{1}{2(1-\nu)}\cdot\frac{1}{1+\alpha}\left[1-2\nu+\frac{1+2\alpha}{1+\alpha}\right] \quad (64)$$

$$S_{1122}=\frac{1}{2(1-\nu)}\cdot\frac{\alpha}{1+\alpha}\left[2\nu-\frac{1}{1+\alpha}\right] \quad (65)$$

$$S_{2211}=\frac{1}{2(1-\nu)}\cdot\frac{1}{1+\alpha}\left[2\nu-\frac{\alpha}{1+\alpha}\right] \quad (66)$$

$$S_{1212}=S_{1221}=\frac{1}{2(1-\nu)}\cdot\frac{\alpha}{1+\alpha}\left[1-\nu+\frac{1}{\alpha}\left\{\nu-\frac{1}{1+\alpha}\right\}\right] \quad (67)$$

$$S_{2112}=S_{2121}=\frac{1}{2(1-\nu)}\cdot\frac{1}{1+\alpha}\left[1-\nu+\alpha\left\{\nu-\frac{\alpha}{1+\alpha}\right\}\right] \quad (68)$$

$$S_{1133}=\frac{\nu}{(1-\nu)}\cdot\frac{\alpha}{1+\alpha} \quad (69)$$

$$S_{2233}=\frac{\nu}{(1-\nu)}\cdot\frac{1}{1+\alpha} \quad (70)$$

$$S_{1313}=S_{1331}=S_{3113}=S_{3131}=\frac{1}{(1-\nu)}\cdot\frac{\alpha}{1+\alpha} \quad (71)$$

$$S_{2323}=S_{2332}=S_{3223}=S_{3232}=\frac{1}{(1-\nu)}\cdot\frac{1}{1+\alpha} \quad (72)$$

where the interior-point Eshelby tensor $\mathbf{S}$ is clearly unrelated to the point $\mathbf{x}$ located outside the inclusion and depends solely on Poisson's ratio of the matrix and the aspect ratio α of the elliptic cylindrical inclusion. Note that the derived components of the interior-point Eshelby tensor given in Eqs. (63)–(71) are identical with those given in Mura [11].

As a special case, by letting $\alpha=1$ in Eqs. (18)–(62), the Eshelby tensor for an elliptic cylindrical inclusion is reduced to the Eshelby tensor for a circular cylindrical inclusion as

$$\hat{G}_{ijkl}=\frac{\rho^2}{8(1-\nu)}F_{ijkl}[8(4\rho^2-3),4(\nu-\rho^2),4(1-\rho^2),4(\rho^2-2\nu-1),4\nu-2+\rho^2,2-4\nu+\rho^2] \quad (73)$$

$$\hat{G}_{ij33}=\frac{\nu\rho^2}{2(1-\nu)}[\delta_{ij}-2\hat{n}_i\hat{n}_j],\quad \hat{G}_{3i3j}=\frac{\rho^2}{4}[\delta_{ij}-2\hat{n}_i\hat{n}_j] \quad (74)$$

where the exterior-point Eshelby tensor for a circular cylindrical inclusion is of minor symmetry; that is, $\hat{G}_{ijkl}=\hat{G}_{jikl}=\hat{G}_{ijlk}$. In addition, the fourth-rank tensor $\mathbf{F}$ is defined as

$$F_{ijkl}[B_m]=B_1\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_l+B_2(\delta_{ik}\hat{n}_j\hat{n}_l+\delta_{il}\hat{n}_j\hat{n}_k+\delta_{jk}\hat{n}_i\hat{n}_l+\delta_{jl}\hat{n}_i\hat{n}_k)+B_3\delta_{ij}\hat{n}_k\hat{n}_l+B_4\delta_{kl}\hat{n}_i\hat{n}_j+B_5\delta_{ij}\delta_{kl}+B_6(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \quad (75)$$

3 Concluding Remarks

This study introduces the newly derived exterior-point Eshelby tensor for an elliptic cylindrical inclusion. With the help of

I-integrals expressed by Mura [11] and the outward unit normal vector introduced by Ju and Sun [10], the closed form solution of the exterior-point Eshelby tensor for an elliptic cylindrical inclusion is derived. The proposed closed form of the exterior-point Eshelby tensor for an elliptic cylindrical inclusion is more explicit than that given in Mura [11], which is rough and unfinished. As a special case, the Eshelby tensor for an elliptic cylindrical inclusion can be reduced to the Eshelby tensor for a circular cylindrical inclusion.

The closed form Eshelby tensor presented in this study can contribute to micromechanics-based analysis of composites with elliptic cylindrical inclusions. In a further study, a micromechanics-based constitutive model that takes into account the Eshelby tensor for an elliptic cylindrical inclusion will be developed to predict the behavior of unidirectional composites containing elliptic cylindrical inclusions.

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